ASYMPTOTIC STABILITY OF THE EQUILIBRIA OF GYROSCOPIC SYSTEMS WITH PARTIAL DISSIPATION

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Mechanical systems acted on by gyroscopic forces with incomplete dissipation are investigated. One of the ways of arriving at such systems is by investigating the stability of steady-state motions. Conditions whose fulfillment implies that the dissipative function ensures asymptotic stability, are derived. An example is considered.

1. Asymptotic stability of the equilibria of gyroscopic systems. Let a holonomic system with steady-state constraints and a force function not explicitly dependent on time, be acted on by the gyroscopic forces Γ_i and by the dissipative forces F_i of the form

$$\Gamma_i = \sum_{j=1}^{n} \gamma_{ij} q_j^{\cdot}, \qquad F_i = \frac{\partial F}{\partial q_i^{\cdot}} \qquad (\gamma_{ij} = -\gamma_{ji} = \mathrm{const}), \quad (i = 1, \ldots, n)$$

where q_i are generalized coordinates and F is a negative definite quadratic form of generalized velocities,

$$F = -\frac{1}{2} \sum_{i,j=1}^{n} \beta_{ij} q_i^{\cdot} q_j^{\cdot} \qquad (\beta_{ij} = \mathrm{const})$$

Let us introduce the Lagrangian

$$L = T + \sum_{i,j=1}^{n} d_{ij}' q_i q_j^{\cdot} + U \qquad (d_{ij}' - d_{ji}' = \gamma_{ij})$$

Here T and U denote the kinetic energy and force function of the system.

The equations of motion can then be written as

$$\frac{d}{dt} \frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = \frac{\partial F}{\partial q_i^{\cdot}} \tag{1.1}$$

We assume that the system is in the equilibrium state $q_i = 0$. Eqs. (1.1) are then equations of perturbed motion.

The first-approximation Eqs. are

$$\frac{d}{dt}\left(\frac{\partial \delta^2 L}{\partial q_i^{\cdot}}\right) - \frac{\partial \delta^2 L}{\partial q_i} = \frac{\partial F}{\partial q_i^{\cdot}}, \qquad \delta^2 L = \delta^2 T + \sum_{i,j=1}^{n} d_{ij}' q_i q_j^{\cdot} + \delta^2 U \tag{1.2}$$

Here $\delta^2 T$ and $\delta^2 U$ are the sets of the second order terms in the expansions of the kinetic energy and force function.

Let $\delta^2 U$ be a negative definite quadratic form q_i. We know [1] that in this case the equilibrium position is stable.

By virtue of Eqs. (1.2) we have

$$d / dt\,(\delta^2 T - \delta^2 U) = F$$

Since $\delta^2 T - \delta^2 U$ is a positive definite function and F is always negative (neither function depends explicitly on time), it follows by the Barbashin-Krasovskii theorem on asymptotic stability [2 and 3] that the equilibrium is asymptotically stable by virtue of the first-approximation system if Eqs. (1.2) do not allow trajectories along which $F \equiv 0$.

Let the rank of the quadratic form F be p. We then have the representation

$$F = -\tfrac{1}{2}\ [(\varphi_1')^2 + \ldots + (\varphi_p')^2]$$

where ϕ_i' denote certain linearly independent forms of generalized velocities,

$$\varphi_i' = \sum_{j=1}^{n} c_{ij} q_j^{\cdot}, \qquad c_{ij} = \mathrm{const} \qquad (i = 1, \ldots, p)$$

If these exist trajectories along which $F \equiv 0$, then on these trajectories

$$\varphi_i' = 0, \qquad \frac{\partial F}{\partial q_l^{\cdot}} = \sum_{i=1}^{p} c_{li} \varphi_i' = 0 \tag{1.3}$$

so that

$$\frac{d}{dt} \frac{\partial \delta^2 L}{\partial q_i^{\cdot}} - \frac{\partial \delta^2 L}{\partial q_i} = 0 \qquad (i = 1, \ldots, n) \tag{1.4}$$

The characteristic equation of system (1.4) has no zero roots, so that Eqs. (1.3) and (1.4) imply that

$$\varphi_i = \sum_{j=1}^{n} c_{ij} q_j = 0 \qquad (i = 1, \ldots, p) \tag{1.5}$$

Let us rewrite Eqs. (1.4) in the Hamiltonian form,

$$q_i^{\cdot} = \partial H / \partial p_i, \qquad p_i^{\cdot} = -\,\partial H / \partial q_i \qquad (H = \delta^2 T - \delta^2 U) \tag{1.6}$$

The function H is the positive definite quadratic form in $q_1, \ldots, q_n^{\cdot}; p_1, \ldots, p_n^{\cdot}$. In this case (according to [4 and 5]) the roots of the characteristic equation are all purely imaginary, i.e. of the form $\pm \lambda_k i$; moreover, there exists a canonical transformation $q_i = q_i\ (x_i, y_i)$, $p_i = p_i\ (x_i, y_i)$ which transforms H to the normal form

$$H = \frac{1}{2} \sum_{i=1}^{n} [(y_i^{\cdot})^2 + \lambda_i^2 x_i^2]$$

The above transformation is linear; its coefficients are constant and (in the general case) complex. Let us express it in matrix form as

$$\begin{Vmatrix} q \\ p \end{Vmatrix} = B \begin{Vmatrix} x \\ y \end{Vmatrix}, \qquad B = \begin{Vmatrix} \partial q_i / \partial x_j & \partial q_i / \partial y_j \\ \partial p_i / \partial x_j & \partial p_i / \partial y_j \end{Vmatrix} \tag{1.7}$$

In the normal variables x_i, y_i Eqs. (1.6) become

$$x_i^{\cdot} = y_i, \qquad y_i^{\cdot} = -\,\lambda_i^2 x_i \tag{1.8}$$

Hence

$$x_i^{\cdot\cdot} = -\,\lambda_i^2 x_i \tag{1.9}$$

Eqs. (1.5) with allowance for (1.7) and (1.8) can be written as

$$D_1 X + D_2 X^{\cdot} = 0 \tag{1.10}$$

$$D_1 = \| d_{kj} \| = C \| \partial q_i / \partial x_j \|, \quad D_2 = \| d_{k,\,n+j} \| = C \| \partial q_i / \partial y_j \| \quad (k = 1, \ldots, p;\ i, j = 1, \ldots, n)$$

Let $\pm \lambda_1 i, \ldots, \pm \lambda_k i$ be all the roots of the characteristic equations with the multiplicities $n_1, \ldots, n_k$. We shall denote the quantity A corresponding to the root $\lambda_l i$ by the symbol

$A(\lambda_l)$. Eqs. (1.9) and (1.10) in this notation become

$$(x_i(\lambda_l))^{\cdot\cdot} = -\lambda_l^2 x_i(\lambda_l) \qquad (i = 1, \ldots, n) \tag{1.11}$$

$$\sum_{l=1}^{k} \sum_{j=\nu_{l-1}}^{\nu_l} [d_{mj} x_j(\lambda_l) + d_{m,\, n+j}(x_j(\lambda_l))^{\cdot}] = 0 \quad (\nu_l = n_1 + \ldots + n_l;\ m = 1, \ldots, p) \tag{1.12}$$

Any solution of Eqs. (1.11) and (1.12) is of the form

$$x_j(\lambda_l) = u_j(\lambda_l)\exp(\lambda_l it) + v_j(\lambda_l)\exp(-\lambda_l it) \qquad (i = 1, \ldots, n)$$

Substituting this solution into Eqs. (1.12) and recalling that the functions $\exp(\lambda_k it)$ and $\exp(\lambda_m it)$ $(\lambda_k \neq \lambda_m)$ are linearly independent, we obtain the following necessary and sufficient conditions for the existence of a nonzero solution of Eqs. (1.11) and (1.12):

$$r[D(\lambda_l)] = n_l, \quad D(\lambda_l) = \| d_{mi} + \lambda_l i d_{m,\, n+j} \| \quad (m = 1, \ldots, p;\ i = \nu_{l-1} + 1, \ldots, \nu_l) \tag{1.13}$$

On fulfillment of Eqs. (1.13), system (1.4) disallows nonzero trajectories along which $F = 0$, and the equilibrium is asymptotically stable by virtue of the first approximation system, and hence by virtue of complete system (1.1).

If $r[D(\lambda_l)] < n_l$, then (for example) in the case where the rank p is smaller than n_l, Eqs. (1.11) and (1.12) allow a nonzero solution, and the first approximation implies the absence of asymptotic stability.

Taking into account (1.7), we can express the matrix $D(\lambda_l)$ in the form

$$D(\lambda_l) = CM(\lambda_l), \qquad M(\lambda_l) = \| \partial q_m / \partial x_j(\lambda_l) + i\lambda_l \partial q_m / \partial y_j(\lambda_l) \|$$

If $r[C]$ is equal to n, then Eqs. (1.5), and therefore Eqs. (1.12), allow a trivial solution only, so that Eqs. (1.13) are fulfilled. This means that $r[M(\lambda_l)] = n_l$.

Computing the minors of order n_l of the matrices $D(\lambda_l)$, we obtain certain polynomials P_i in the variables c_{ij} of degree not exceeding n_l. Since $r[M(\lambda_l)] = n_l$, none of the polynomials P_i vanish identically. Hence, there exist real c_{ij} such that all $P_i \neq 0$. In this case the equilibrium is asymptotically stable.

We have thus proved the following statement.

Theorem. Let the expansion of the force function of the gyroscopic system in the neighborhood of the equilibrium position begin with a negative definite quadratic form. In the case where the largest multiplicity s of a root of the characteristic equation of system (1.4) does not exceed the rank p of the dissipative function, the equilibrium is asymptotically stable by virtue of the complete system of equations, provided the rank of each matrix

$$D(\lambda_l) = \begin{Vmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdots & \cdot \\ c_{p1} & \cdots & c_{pn} \end{Vmatrix} \begin{Vmatrix} \dfrac{\partial q_m}{\partial x_j(\lambda_l)} + \lambda_l i \dfrac{\partial q_m}{\partial y_j(\lambda_l)} \end{Vmatrix} \quad \begin{matrix} (m, j = 1, \ldots, n) \\ (l = n_1 + \ldots + n_{l-1} + 1, \ldots, n_1 + \ldots + n_l) \end{matrix}$$

is equal to the multiplicity of this root. This is the necessary and sufficient condition for asymptotic stability in the first approximation. The addition of any dissipative forces with a dissipative function of rank smaller than s does not render the equilibrium stable by virtue of the first approximation. On the other hand, there always exists a dissipative function of rank s such that the equilibrium is asymptotically stable.

2. Asymptotic stability of steady-state motions. Let us consider a holonomic mechanical system whose kinetic energy T and force function U do not depend explicitly on time and on the last k generalized coordinates $q_{n-k+1}, \ldots, q_n$. We assume that the indices r and s vary from 1 to $n-k$, and the indices m and l from $n-k+1$ to n. Let the system be acted on by dissipative forces with the dissipative function

$$F = -\frac{1}{2} \sum_{i,j=n-p+1}^{n} \beta_{ij} q_i^{\cdot} q_j^{\cdot} \qquad (p \geqslant k)$$

negative-definite with respect to its variables, and by some constant forces $F_{n-k+1}, \ldots, F_n$ such that the system allows steady-state motion [6],

$$q_r = 0, \qquad q^{\cdot}_m = q^{\cdot}_{m0} \tag{2.1}$$

It is shown in [7] that when the quadratic part of the expansion of $T - U$ in the neighborhood of steady-state motion (2.1) is positive definite with respect to the variations of the coordinates and velocities, motion (2.1) is asymptotically stable if a system with the kinetic energy T' given by

$$T' = \frac{1}{2}\left[\sum_{r,s} a_{rs} q_r^{\cdot} q_s^{\cdot} + \sum_{r,l} a_{rl} q_r^{\cdot} q_{l0}^{\cdot}\right]$$

and the force function U' of the form

$$U' = U + \sum_{m,l} a_{ml} q_{m0}^{\cdot} q_{l0}^{\cdot}$$

does not have motions in the neighborhood of the equilibrium $q_r = 0$ such that

$$\sum_s a_{ms} q_s^{\cdot} + \sum_l a_{ml} q_{l0}^{\cdot} = 0, \qquad q_{n-p+1} = \ldots = q_{n-k} = 0 \quad (p \geqslant k) \tag{2.2}$$

The variations of Eqs. (2.2) can be written as

$$\sum_r (a_{mr})_0\, q_r^{\cdot} + \sum_{l,s} \left(\frac{\partial a_{ml}}{\partial q_s}\right)_0 q_s q_{l0}^{\cdot} = 0, \qquad q_{n-p+1} = \ldots = q_{n-k} = 0 \tag{2.3}$$

The first-approximation equations for a system with the Lagrangian $L' = T' + U'$ become

$$\frac{d}{dt}\frac{\partial \delta^2 L'}{\partial q_r^{\cdot}} - \frac{\partial \delta^2 L'}{\partial q_r} = 0 \tag{2.4}$$

The reasoning of Section 1 applies, in slightly altered form, to Eqs. (2.3) and (2.4). The conditions for asymptotic stability analogous to (1.13) are

$$r\,[D(\lambda_p')] = n_l', \qquad D(\lambda_l') = C' \left\| \begin{matrix} \partial q_i / \partial x_j (\lambda_l') + i\lambda_l' \partial q_i / \partial y_j (\lambda_l') \\ -(\lambda_l')^2\, \partial q_i / \partial y_j (\lambda_l') + i\lambda_l' \partial q_i / \partial x_j (\lambda_l') \end{matrix} \right\|$$

Here $i\lambda_l'$ is the root of the characteristic equation of system (2.4), n_l' is the multiplicity of this root, C' is the matrix of coefficients of Eqs. (2.3), and x_i', y_i' are the normal variables of Eqs. (2.4).

Let the dissipative function F be independent of the cyclic coordinates. The equations of motion for the noncyclic coordinates are the Routh Eqs.

$$\frac{d}{dt}\frac{\partial R}{\partial q_r^{\cdot}} - \frac{\partial R}{\partial q_r} = \frac{\partial U}{\partial q_r} + \frac{\partial F}{\partial q_r^{\cdot}} \qquad (R = R_2 + R_1 - R_0) \tag{2.5}$$

Here R_2 is the positive definite quadratic form of the velocities, R_1 is the linear form of the velocities, and R_0 depends on the coordinates only.

Eqs. (2.5) are of a form analogous to (1.1), so that if the expansion of $U - R_0$ begins with a negative definite quadratic form of the variables q_r, then the results of Section 1 can be applied to the investigation of asymptotic stability.

R e m a r k. The proof of Lemma 3.1 in [8] contains an inaccuracy. The lemma should be altered to read L e m m a 3.1. If the introduced dissipation renders the equilibrium asymptotically stable, then none of the coefficients $\nu_{k+1}^2, \ldots, \nu_n^2$ are equal to any one of the numbers $\lambda_1^2, \ldots, \lambda_n^2$ although some of $\nu_{k+1}^2, \ldots, \nu_n^2$ may coincide.

The arguments of [8] which follow this lemma remain valid only if all the new frequencies $\nu_{k+1}^2, \ldots, \nu_n^2$ are distinct.

3. Example. Let us consider a mechanical system in the form of a solid body with a pendumlum suspended from its point O (i.e. from its center of mass). We assume that the system is not acted on by any external forces. Let $Ox_1x_2x_3$ be a stationary coordinate system, and $Oy_1y_2y_3$ a movable coordinate system with axes directed along the principal axes of inertia of the body. The pendulum is suspended in such a way that the motion occurs in the plane Oy_1y_2. We denote the angle between the pendulum and the negative direction of the Oy_2-axis by α. Acting at the axis of the pendulum suspension are the moment of viscous friction forces $k\alpha^{\cdot}$ and the moment of elastic forces $\varkappa\alpha$.

Let us introduce the following notation: x_1', x_2', x_3' are the coordinates of the center of mass of the system with respect to $Ox_1x_2x_3$; ϕ, ψ, θ are the Euler angles defining the position of $Oy_1y_2y_3$; p_1, p_2, p_3 are the projections of the instantaneous angular velocity of the body on the axes y_1, y_2, y_3; A_1, A_2, A_3 are the moments of inertia of the body with respect to y_1, y_2, y_3; l is the length of the pendulum; M is the mass of the body; m is the mass of the pendulum.

The coordinates ψ, x_1', x_2', x_3' are cyclic, and the equations of motion can be written in the form (2.5).

The steady-state motion under investigation is described by Eqs.

$$\theta = \pi/2, \quad \varphi = 0, \quad \alpha = 0, \quad \psi^{\cdot} = \psi_0^{\cdot} = \omega, \quad (x_i') = (x_i')_0 \qquad (i = 1, 2, 3) \quad .$$

Assuming that the cyclic impulses are not perturbed, we set

$$\theta = \pi / 2 + \xi_1, \quad \varphi = \xi_2, \quad \alpha = \xi_3 \tag{3.1}$$

for the perturbed motion.

The conditions of negative definiteness of $\delta^2(U - R_0)$ are given by the inequalities

$$\omega^2 (B_2 - B_3) > 0, \quad \omega^2 (B_2 - B_1) > 0, \quad (B_2 - B_1)(\varkappa - a\omega^2) - a^2\omega^2 > 0$$

$$B_1 = A_1 + a, \quad B_2 = A_2 + a, \quad B_3 = A_3 + a, \quad a = Mml^2 / M + m$$

The dissipative function is of the form $F = -\frac{1}{2}k(\xi_3^{\cdot})^2$.

Setting $\xi_3 = \xi_3^{\cdot} = 0$ for the perturbations of the noncyclic coordinates in the Routh equations, we obtain in the first approximation a system of the form (1.4) with the function

$$\delta^2 L = \tfrac{1}{2}\,[B_1(\xi_1^{\cdot})^2 + B_3(\xi_2^{\cdot})^2 + 2\omega\gamma_{12}\xi_2\xi_1^{\cdot} + (B_3 - B_2)\omega^2\xi_1^2 + (B_1 - B_2)\omega^2\xi_2^2]$$

$$\gamma_{12} = B_1 + B_2 - B_3$$

In addition, Eq.

$$\varphi_1 = \omega\gamma_{12}(\xi_1^{\cdot} + \omega\xi_2) = 0 \tag{3.2}$$

will hold.

In normal variables Eq. (3.2) becomes

$$\gamma_{12}\left\{i\omega\left[\frac{(B_2 - B_1)(B_2 - B_3)}{B_1 B_2}\right]^{1/2}\left(-\frac{B_2}{B_1 - B_2} - \frac{1}{4B_1}\right)x_2 + \left(-\frac{B_2}{B_1 - B_2} + \frac{1}{4B_1}\right)y_2\right\} = 0$$

This implies that $D(\lambda_1) = 0$ and that condition (1.13) is not fulfilled. Hence, the steady-state motion is not asymptotically stable by virtue of the first-approximation system.

Differentiating Eq. (3.2) with respect to time, we find by virtue of the equations of perturbed motion that

$$\gamma_{12}(B_3 - B_2)\omega(\xi_2^{\cdot} - \omega\xi_1) = 0$$

In the first approximation

$$p_1 = \omega\xi_2 + \xi_1^{\cdot}, \quad p_2 = \omega + \eta^{\cdot}, \quad p_3 = \xi_2^{\cdot} - \omega\xi_1 \qquad (\eta^{\cdot} = 0)$$

Hence, on the trajectories along which $F \equiv 0$ we have in the first approximation

$$p_1 = p_3 = 0, \quad p_2 = \mathrm{const}$$

The proof of the Barbashin-Krasovskii theorem mentioned above implies that if the equations allow trajectories along which $F \equiv 0$, then the motion tends asymptotically either to zero, or to one of the indicated trajectories. Hence, in the first approximation for $\omega\gamma_{12} \neq 0$ all the motions (3.1) tend asymptotically to rotation of the system about the axis y_2 which has a constant direction in space.

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BIBLIOGRAPHY

1. Chetaev, N.G., Stability of motion. "Nauka" Press, 1965.
2. Krasovskii, N.N., Some Problems of Motion Stability Theory. Fizmatgiz, 1960.
3. Barbashin, E.A. and Krasovskii, N.N., Stability of motion in the whole. Dokl. Akad. Nauk SSSR, 1952.
4. Whittaker, E.T., Analytical Dynamics. Gostekhizdat, 1956.
5. Synge, J.L., Classical Dynamics. Fizmatgiz, 1963.
6. Pozharitskii, G.K., The stability of dissipative systems. PMM Vol. 21, No. 4, 1957.
7. Pozharitskii, G.K., On asymptotic stability of equilibria and stationary motions of mechanical systems with partial dissipation. PMM Vol. 25, No. 4, 1961.
8. Pozharitskii, G.K., Characteristic damping exponents for vibrations of mechanical systems with partial dissipation. PMM Vol. 29, No. 5, 1965.

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